FALL 2024: MATH 790 EXAM 1

Throughout this exam, V will denote a vector space over the field F. Unless stated otherwise, we do not assume that V is finite dimensional. Each problem is worth 10 points. You may use the Daily Update, your notes from class or any homework you have done, **but you may not use any other resources**, including your book, any other book, any information taken from the internet, nor may you consult with any students or professors, other than your Math 790 professor. You may freely use basic linear algebra facts presented in a first course on linear algebra, but you **may not** use any advanced linear algebra facts not presented in class. Please upload your solutions in **pdf format** - and no other format - to Canvas by 5pm on Monday, October 7. **Good luck on the exam!**

1. Let K be a subfield of F. We can regard F as a vector space over K, using the addition in F as vector addition and multiplication of elements in K times elements in F (via the given multiplication in F) as scalar multiplication. Note that any vector space V over F is automatically a vector space over K. Assume that V is finite dimensional over F and F is finite dimensional as a vector space over K. Find and prove a formula that expresses $\dim_K(V)$ in terms of $\dim_F(V)$ and $\dim_K(F)$. Hint: It might be helpful to first consider thet case $K = \mathbb{R}$ and $F = \mathbb{C}$.

Solution. Let $\lambda_1, \dots, \lambda_n$ be a basis for the field F regarded as a vector space over the field K and $v_1, \dots, v_m$ be a basis for the vector space V over F. We will show that the set $B := \{\lambda_i v_j\}_{1\leq i\leq n}^{1\leq j\leq m}$ is a basis for V over K. Upon doing so, it will follow that $\dim_K(V) = \dim_K(F)\cdot \dim_F(W)$.

Now, take $v \in V$. Then we may write $v = \alpha_1 v_1 + \cdots + \alpha_m v_m$, with each $\alpha_i \in F$. On the other hand, we may write each $\alpha_i = \beta_{i,1}\cdot\lambda_1 + \cdots + \beta_{i,n}\cdot\lambda_n$, with each $\beta_{i,j} \in K$. Substituting these latter relations into the first equation, we get

$$v = (\beta_{1,1}\lambda_1 + \cdots + \beta_{1,n}\lambda_n)v_1 + \cdots + (\beta_{m,1}\lambda_1 + \cdots + \beta_{m,n}\lambda_n)v_m.$$

Expanding out this last identity we see that $v = \sum_{i,j}\beta_{j,i}\cdot(\lambda_i v_j)$, which shows that the set B spans V over K.

Finally, to see that the set B is linearly independent over K, suppose we have a dependence relation $\sum_{1\leq i\leq n}^{1\leq j\leq m}\gamma_{j,i}\cdot(\lambda_i v_j) = 0$, with each $\gamma_{j,i} \in K$. Then, we may rewrite this equation as

$$0 = (\gamma_{1,1}\lambda_1 + \cdots + \gamma_{1,n}\lambda_n)v_1 + \cdots + (\gamma_{m,1}\lambda_1 + \cdots + \gamma_{m,n}\lambda_n)v_m.$$

Note that the coefficients of the v_i in this last equation belong to F. Thus,

$$0 = \gamma_{1,1}\lambda_1 + \cdots + \gamma_{1,n}\lambda_n = \cdots = \gamma_{m,1}\lambda_1 + \cdots + \gamma_{m,n}\lambda_n,$$

since the v_j are linearly independent over F. Since the λ_j are linearly independent over K, it follows that all $\gamma_{j,i}$ are zero, which gives what we want. The proof is now complete. □

2. Let W be a proper subspace of the vector space V. First prove that there exists a subspace $U \subseteq V$ that is maximal with respect to the property that $W \cap U = 0$, and then show that $V = W \oplus U$.

Solution. Let W be a subspace of V. We need to check that the hypotheses of Zorn's Lemma apply to the set $S := \{W' \subseteq V \mid W' \text{ is a subspace of } V \text{ and } W \cap W' = 0\}$. Note, the partial order on S is just inclusion of subspaces. For this, we take a chain $\{W_i'\}_{i\in I}$ in S, i.e., a totally ordered collection of subspaces $\{W_i'\}$ of V with $W \cap W_i' = 0$, for all i. To find an upper bound for the chain in S, we consider $W_0' := \bigcup_{i\in I} W_i'$. As in class, a union of totally ordered subspaces of V is a subspace of V, so W_0' is a subspace of V. Let $x \in W_0' \cap W$. Since $x \in W_0'$, $x \in W_i'$, some i. Thus, $x \in W_i' \cap W = 0$, since $W_i' \in S$. Thus, $W_0' \cap W = 0$, so $W_0' \in S$. Therefore, W_0' is an upper bound in S for the given chain. It follows from Zorn's Lemma that S has a maximal element, say U.

To see that $V = W \bigoplus U$, we just have to show $V = W + U$, since we already have $W \cap U = 0$. Take $v \in V$. If $v \in U$, then $v \in W + U$. Suppose $v \notin U$. Then $U_0 := U + \langle v\rangle$ is a subspace of V properly containing U, so, by the maximality of U, $U_0 \cap W \neq 0$. Let $0 \neq w \in U_0 \cap W$. Then $w = u + \alpha v$, for some $u \in U$, $\alpha \in F$ and $w \in W$. Notice that $\alpha \neq 0$, otherwise, w would be a non-zero vector in $W \cap U$. It follows that $v = \frac{-1}{\alpha}(w - u)$, so that $v \in W + U$, as required. Thus, $V = W + U$, so $V = W \oplus U$. □

3. Suppose $\dim(V)$ equals $n > 0$. Prove there cannot exist a chain of subspaces $(0) \subsetneq W_1 \cdots \subsetneq W_n \subsetneq V$. Conclude that if $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is an ascending chain of subspaces of V, then there exists $n_0 \geq 1$ such that $U_s = U_{n_0}$, for all $s \geq n_0$. In particular, show that if $T \in \mathcal{L}(V, V)$, there exists $1 \leq r \leq n$ such $\ker(T^j) = \ker(T^r)$, for all $j \geq r$.

Solution. The first two statement follow, since given subspaces $A \subseteq B \subseteq V$, $A \subsetneq B$ if and only if $\dim(A) < \dim(B)$. For the third statement, suppose $\text{kernel}(T^r) = \text{kernel}(T^{r+1})$, for some $r \geq 1$. Then $\text{kernel}(T^{r+j}) = \text{kernel}(T^r)$, for all $j \geq 1$. To see this, induct on j. The case $j = 1$ is true by assumption. Suppose the result is true for $j-1$. We want to show kernel(T^j) = kernel(T^r). Take $v \in$ kernel(T^j), so that $T^j(v) = 0$. Then $T^{j-1}(T(v)) = 0$, so $T(v) \in$ kernel(T^{j-1}). Thus, $T(v) \in$ kernel(T^r). Therefore $T^r(T(v)) = 0$, so $T^{r+1}(v) = 0$. By hypothesis, $T^r(v) = 0$, which is what we want. Now, because we cannot have more than n strict containment of subspaces in V, we must have kernel(T^r) = kernel(T^{r+1}), for sone $1 \leq r \leq n$. □

4. Using the ideas of the previous problem, show that if $\dim(V) = n$, and $T \in \mathcal{L}(V,V)$, then $V = \ker(T^n) \oplus \operatorname{im}(T^n)$.

Solution. We first show that for all $j \geq 1$, image(T^{r+j}) = image$(T^r) = \cdots$, for the r found in the previous problem such that such that kernel(T^r) = kernel(T^{r+j}). For this, since kernel(T^{r+j}) = kernel(T^r), these spaces have the same dimension. By the Rank Plus Nullity theorem, image(T^{r+j}) and image(T^r) have the same dimension. Since image$(T^{r+j}) \subseteq$ image(T^r), these subspaces must be equal.

Now, we note that kernel$(T^n) \cap$ image$(T^n) = 0$. To see this, take $v \in$ kernel$(T^n) \cap$ image(T^n), so that $T^n(v) = 0$ and $v = T^n(v')$, for some $v' \in V$. Then $0 = T^n(v) = T^{2n}(v')$, so that $v' \in$ kernel(T^{2n}) = kernel(T^r) = kernel(T^n). Thus, $v = T^n(v') = 0$. To see that V = kernel(T^n) + image(T^n), take $v_1, \ldots, v_s$ a basis for kernel(T^n) and $u_1, \ldots, u_t$ a basis for image(T). By the Rank Plus Nullity Theorem, $s + t = n$. If we show that $v_1, \ldots, v_2, u_1, \ldots, u_t$ are linearly independent, then they form a basis for V. Thus,

$$V = \text{Span}\{v_1, \ldots, v_s\} + \text{Span}\{u_1, \ldots, u_t\} = \text{kernel}(T^n) + \text{image}(T^n),$$

which gives what we want. Suppose $a_1v_1 + \cdots + a_sv_2 + b_2u_1 + \cdots + b_tu_t = 0$, for $a_i, b_j \in F$. Then

$$a_1v_1 + \cdots + a_sv_s = -b_1u_1 - \cdots - b_tu_t.$$

This vector belongs to kernel$(T^n) \cap$ image$(T^n) = 0$, so we have $a_1v_1 + \cdots + a_sv_s = 0$ and $0 = -b_1u_1 - \cdots - b_tu_t$.. Since the v_i are linearly independent, each $a_i = 0$ and since the u_j are independent, each $b_j = 0$. This shows $v_1, \ldots, u_t$ are linearly independent, which is what we want. □

5. Let V be a vector space of dimension n over the field F.
   (i) Prove that the vector spaces $\mathcal{L}(V,V)$ and $\mathrm{M}_n(F)$ are isomorphic.
   (ii) Using the Cayley-Hamilton theorem for matrices, prove that $\chi_T(T) = 0$, for $T \in \mathcal{L}(V,V)$.
   (iii) For $f(x) \in [x]$, with s the degree of $f(x)$, prove that $|xI_s - C(f(x))| = f(x)$, where $C(f(x))$ is the companion matrix of $f(x)$. In other words $\chi_{C(f(x))}(x) = f(x)$.

Solution. For part (i), one can either note that each space in question has dimension n^2, and then use the fact that vector spaces of the same dimension are isomorphic, or one can explicitly exhibit an isomorphism. In the first case, fix a basis $B = \{v_1, \ldots, v_n\}$ for V. Define $T_{ij} : V \to V$ by $T_{ij}(v_i) = v_j$ and $T_{ij}(v_k) = 0$, for $k \neq i$. One shows that $\{T_{ij}\}$ is a basis for $\mathcal{L}(V,V)$. Similarly, one defines $E_{ij} \in \mathrm{M}_n(F)$ to be the matrix with (i,j) entry equal to 1 and all other entries equal to zero. Then $\{E_{ij}\}$ is a basis for $\mathrm{M}_n(F)$. Both bases have n^2 elements. In the second case one defines $\phi : \mathcal{L}(V,V) \to \mathrm{M}_n(F)$ by $\phi(T) = [T]_B^B$ and easily checks that ϕ is an isomorphism.

For (ii), let $T \in \mathcal{L}(V,V)$, fix a basis $B \subseteq V$ and set $A := [T]_B^B$. Then, by definition, $\chi_A(x) = \chi_T(x)$. To see that $\chi_T(T) = 0$, we first need an observation: Suppose $p(x) \in F[x]$. Then $[p(T)]_B^B = p([T]_B^B)$. This observation follows easily from two facts: $[ST]_B^B = [S]_B^B[T]_B^B$, for all $S, T \in \mathcal{L}(V,V)$ and $[aS + bT]_B^B = a[S]_B^B + b[T]_B^B$, for all $a, b \in F$. Thus,

$$0 = \chi_A(A) = \chi_T(A) = \chi_T([T]_B^B) = [\chi_T(T)]_B^B,$$

which implies that $\chi_T(T) = 0$, since a linear transformation is zero, if it is zero on all elements of a basis.

For part (iii), set $C := C(f(x))$. Then $\chi_C(x) = \begin{vmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ 0 & -1 & x & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & x + a_{n-1} \end{vmatrix}$. Expanding along the first row, we get

$$\chi_C(x) = x \cdot \begin{vmatrix} x & 0 & 0 & \cdots & 0 & a_1 \\ -1 & x & 0 & \cdots & 0 & a_2 \\ 0 & -1 & x & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & x + a_{n-1} \end{vmatrix} + (-1)^{n+1}a_0 \cdot \begin{vmatrix} -1 & x & 0 & \cdots & 0 \\ 0 & -1 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix},$$

where the determinant on the left is the determinant one calculates for the characteristic polynomial of $C(g(x))$, for $g(x) = x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$, which, by induction on n is $g(x)$. The determinant on the right is an

$(n-1)\times(n-1)$ upper triangular matrix with -1 down the diagonal, and thus equals, $(-1)^{n-1}$. It follows that $\chi_C(x) = xg(x) + a_0 = f(x)$, as required. ☐

6. Let $W_1, \ldots, W_n \subseteq V$ be subspaces and assume $V = W_1 + \cdots + W_n$. Prove that there exists $r \leq n$ and subspaces $U_{i_1} \subseteq W_{i_1}, \ldots, U_{i_r} \subseteq W_{i_r}$ such that $V = U_{i_1} \bigoplus \cdots \bigoplus U_{i_r}$. Note: we are not assuming that V is finite dimensional. Hint: Consider the $n = 2$ case first. Hint 2: This can be done using bases, but there is an easier way without appealing to bases.

Solution. In order to prove that there exist subspaces $U_1 \subseteq W_1, \ldots, U_n \subseteq W_n$ such that $V = U_1 \oplus \cdots \oplus U_n$, we first discard any redundant W_i. In other words, if some $W_j \subseteq W_1 + \cdots + W_{j-1} + W_{j+1} + \cdots + W_n$, we may discard it and take $U_j = 0$. Thus, after finitely many such steps, we may begin again (after a change in notation) and assume that there are no redundancies, i.e., V is not the sum of any proper subset of the $W_i's$. We now proceed by induction on n, starting with the case $n = 2$.

If $n = 2$, write $W_2 = (W_1 \cap W_2) \oplus U_2$, for some $U_2 \subseteq W_2$. We can do this since every subspace of a vector space has a complement, by problem 2. Then

$$V = W_1 + W_2 = W_1 + (W_1 \cap W_2) + U_2 = W_1 + U_2.$$

On the other hand, $W_1 \cap U_2 = (W_1 \cap W_2) \cap U_2 = 0$, thus, $W_1 + W_2 = W_1 + U_2 = W_1 \oplus U_2$. Taking $U_1 = W_1$ completes the proof of the case $n = 2$.

For $n > 2$, by induction, we may find $U_1 \subseteq W_1, \ldots, U_{n-1} \subseteq W_{n-1}$, such that for $V_0 := W_1 + \cdots + W_{n-1}$, $V_0 = U_1 \oplus \cdots \oplus U_{n-1}$. On the other hand, by the case $n = 2$ (and its proof), there exists $U_n \subseteq W_n$ such that $V_0 + W_n = V_0 \oplus U_n$. Thus, in particular, $W_1 + \cdots + W_n = V_0 + U_n = U_1 + \cdots + U_{n-1} + U_n$. To see that this latter sum is direct, suppose $u_1 + \cdots + u_{n-1} + u_n = 0$, with each $u_j \in U_j$. We must show each $u_j = 0$. Regarding $u_1 + \cdots + u_{n-1}$ as an element of V_0, by the directness of the sum $V_0 \oplus U_n$, we have $u_n = 0$ and $u_1 + \cdots + u_{n-1} = 0$. This latter equation implies $u_1 = \cdots = u_{n-1} = 0$, since V_0 is the direct sum of $U_1, \ldots, U_{n-1}$. Thus, all $u_j = 0$, and this gives what we want. ☐

7. Let $T : F^n \to F^n$ be a linear transformation, suppose $E \subseteq F^n$ is the standard basis, and write $A = [T]_E^E$. Suppose P is an invertible matrix such that $P^{-1}AP = D$, where D is a diagonal matrix. Let $C_1, \ldots, C_n$ be the columns of P, and set $B := \{C_1, \ldots, C_n\}$. Prove that B is a basis for F^n and $[T]_B^B = D$.

Solution. To see that the set B forms a basis, it suffices to show that $C_1, \ldots, C_n$ are linearly independent. Suppose $a_1C_1 + \cdots + a_nC_n = 0$, with each $a_j \in F$. Then $P \cdot v = 0$, for $v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Since P is invertible, multiplying both sides of this equation by P^{-1} gives $v = 0$, and hence $C_1, \ldots, C_n$ are linearly independent.

For the second statement, we note that $P = [I]_B^E$. Thus,

$$[T]_B^B = [I]_E^B \cdot [T]_E^E \cdot [I]_B^E = P^{-1}AP.$$

☐

8. Let $V := \mathrm{M}_2(\mathbb{C})$ with inner product $\langle A, B\rangle = \mathrm{trace}(A^t \cdot \overline{B})$. Let W denote the subspace spanned by the single matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Find an orthonormal basis for $W^\perp := \{v \in V \mid \langle v, w\rangle = 0, \text{ for all } w \in W\}$.

Solution. We first calculate $W^\perp$. We need the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$0 = \langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle = \mathrm{trace}\{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^t \cdot \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\}.$$

Thus, $\overline{a} + \overline{b} + \overline{c} + \overline{d} = 0$, and hence $a + b + c + d = 0$. Regarding this equation as a system of one equation in four unknowns, the solution space in $\mathbb{C}^4$ has a basis consisting of the vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. It follows that $W^\perp$ is has basis $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We leave the details to you to see that applying the Gram-Schmidt process to these matrices yields $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} & 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{2} \end{pmatrix}$, which form an orthonormal basis for $W^\perp$. ☐

9. Let $C = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{pmatrix}$. Find an orthogonal matrix Q such that $Q^{-1}CQ$ is diagonal. Now, let $T_C : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis of $\mathbb{R}^3$ is C. Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for T.

Solution Outline. $\chi_A(x) = (x+3)^2(x+12)$. E_{-3} is the null space of the matrix $\begin{pmatrix} -1 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}$ which simplifies to $\begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ via elementary row operations. From this, one obtains a basis $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ for E_2. Applying Gram-Schmidt gives the following orthonormal basis for E_{-3}: $u_1 := \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} \frac{-2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{pmatrix}$.

E_{-12} is the nullspace of the matrix $\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}$ which has the unit vector $u_3 = \begin{pmatrix} \frac{1}{3} \\ \frac{-2}{3} \\ \frac{2}{3} \end{pmatrix}$ for a basis. Note that u_1, u_2, u_3 is an orthonormal basis for $\mathbb{R}^3$ and if Q is the matrix with columns u_1, u_2, u_3, Q is an orthogonal matrix and satisfies $Q^{-1}AQ = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 12 \end{pmatrix}$.

For the second statement, just take the columns of Q, and use Problem 7. □

10. Assume that V is an inner product space over $\mathbb{R}$ and $T \in \mathcal{L}(V, V)$ is a linear operator. Show that the conditions (i)-(iv) below are equivalent. Any $T \in \mathcal{L}(V, V)$ satisfying these conditions is called an *isometry*. This is the operator analogue of orthogonal matrix.

(i) $||T(u)|| = ||u||$, for all $u \in V$.
(ii) $\langle T(v), T(w)\rangle = \langle v, w\rangle$, for all $v, w \in V$.
(iii) T takes an orthonormal basis of V to an orthonormal basis.
(iv) $[T]_{B_1}^{B_2}$ is an orthogonal matrix for all orthonormal bases $B_1, B_2 \subseteq V$.

Hint: For (i) implies (ii), try taking $u = v - w$ in (i).

Solution. For (i) implies (ii), it suffices to assume $\langle T(u), T(u)\rangle = \langle u, u\rangle$ for all $u \in V$ and show $\langle T(v), T(w)\rangle = \langle v, w\rangle$, for all $v, w \in V$. On the one hand, $\langle v - w, v - w\rangle = \langle T(v - w), T(v - w)\rangle$, while on the other hand,

$$\langle v - w, v - w\rangle = \langle v, v\rangle - \langle w, v\rangle - \langle v, w\rangle + \langle w, w\rangle$$

and

$$\langle T(v - w), T(v - w)\rangle = \langle T(v), T(v)\rangle - \langle T(w), T(v)\rangle - \langle T(v), T(w)\rangle + \langle T(w), T(w)\rangle.$$

Thus, since we are working over $\mathbb{R}$, $-2\langle v, w\rangle = -2\langle T(v), T(w)\rangle$, and thus, $\langle v, w\rangle = \langle T(v), T(w)\rangle$, which is what we want.

Now suppose (ii) holds. If $\{v_1, \dots, v_n\}$ is an orthonormal basis, then $\langle T(v_i), T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij}$, for all i, j showing that $\{T(v_1), \dots, T(v_j)\}$ is an orthonormal basis.

Suppose (iii) holds and $B_1, B_2 \subseteq V$ are orthonormal bases. We have $[T]_{B_1}^{B_2} = [T]_{B_2}^{B_2} \cdot [I]_{B_1}^{B_2}$, so it suffices to check that $[T]_B^B$ is orthogonal whenever B is an orthonormal basis and $[I]_{B_1}^{B_2}$ is orthogonal since if $A = BC$, and B, C are orthogonal then

$$A^t = (BC)^t = C^t B^t = C^{-1}B^{-1} = (BC)^{-1} = A^{-1},$$

showing A is orthogonal. We use the following identity proven in class: For $v, w \in V$, and $B \subseteq V$ an orthonormal basis, $\langle, v, w\rangle = \langle [v]_B, [w]_B\rangle$, where the first inner product is in V and the second in $\mathbb{R}^n$.

Now suppose $B = \{v_1, \dots, v_n\}$ is an orthonormal basis for V and $A = [T]_B^B$. Let $C_1, \dots, C_n$ denote the columns of A. Then,

$$\langle C_i, C_j\rangle = \langle [T(v_i)]_B, T(v_j)]_B\rangle = \langle T(v_i), T(v_j)\rangle = \delta_{ij},$$

since $\{T(v_1, \dots, T(v_n)\}$ is an orthonormal basis for V. Thus, the columns of A form an orthonormal basis for $\mathbb{R}^n$, showing that A is an orthogonal matrix. Therefore, (iii) implies (iv).

To see that (iv) implies (ii), it is easy to check that one can reverse the argument showing (iii) implies (iv), so we can assume T takes an orthonormal basis to an orthonormal basis. Let $B = \{u_1, \ldots, u_n\}$ be an orthonormal basis for V. Take $u \in V$ and write $u = a_1u_1 + \cdots + a_nu_n$, for $a_j \in \mathbb{R}$. Then

$$\langle u, u\rangle = \langle \Sigma_i a_i u_i, \Sigma_j a_j u_j\rangle = \Sigma_{i,j} a_i a_i \langle u_i, u_j\rangle = a_1^2 + \cdots + a_n^2 = \Sigma_{i,j} a_i a_j \langle T(u_i), T(u_j)\rangle = \langle T(u), T(u)\rangle.$$

Taking square roots we get $||u|| = ||T(u)||$, as required. □

Bonus Problems. Each Bonus problem is worth 10 points. Your solution must be completely, or nearly completely, correct to earn any bonus points.

BP1. Let V be an inner product space over $\mathbb{R}$ and S, T commuting, symmetric on V. Prove that there exists an orthonormal basis B such that $[T]_B^B$ and $[S]_B^B$ are diagonal, i.e., S and T are simultaneously, orthogonally diagonalizable.

Solution. Since T is symmetric, its distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ are in $\mathbb{R}$ and $V = E_{\lambda_1} \bigoplus \cdots \bigoplus E_{\lambda_r}$. Moreover, for all $i \neq j$ any vector in E_{λ_i} is orthogonal to any vector in E_{λ_j}. Thus if B_i is an orthogonal basis for E_{λ_i}, then $B := B_1 \cup \cdots \cup B_r$ is an orthogonal basis for V. Now, take $v \in E_{\lambda_i}$, so that $T(v) = \lambda_i v$. Then

$$T(S(v)) = TS(v) = ST(v) = S(\lambda_i v) = \lambda_i S(v).$$

Thus, $S(v) \in E_{\lambda_i}$. This shows that S takes vectors in E_{λ_i} to vectors in E_{λ_i}. Thus $S_{|_{E_{\lambda_i}}} \in \mathcal{L}(E_{\lambda_i}, E_{\lambda_i})$. It is easy to check that $S_{|_{E_{\lambda_i}}}$ is symmetric. Thus, there exists an orthonormal basis B_i consisting of eigenvectors of $S_{|_{E_{\lambda_i}}}$. By definition, B_i consists of eigenvectors for T. Taking $B = B_1 \cup \cdots \cup B_r$ gives an orthonormal basis for V consisting of eigenvectors for both T and S. □

NOTE. Even though S and T in the previous problem have a common set of eigenvectors, the eigenvalues need not be the same. Indeed, if $p(x)$ is any polynomial in x with coefficients in F, and $T(v) = \lambda v$. One easily checks that T and $p(T)$ commute and $p(T)(v) = p(\lambda)v$.

BP2. Let A denote the $n \times n$ matrix over $\mathbb{R}$ such that every entry of A is 1. Prove that A is diagonalizable.

Solution. The matrix A has rank 1, so that the solution space to the homogenous system of linear equations with coefficient matrix A has dimension $n-1$. In other words, 0 is an eigenvalue of A with geometric multiplicity $n-1$. Thus, the algebraic multiplicity of 0 is at least $n-1$. If the algebraic multiplicity of 0 were n, then we would have to have $\chi_A(x) = x^n$. But A^n is the $n \times n$ matrix whose entries are all n, and thus, we cannot have $\chi_A(x) = x^n$, by the Cayley-Hamilton theorem. Therefore, $\chi_A(x) = x^{n-1}g(x)$, where $g(x)$ is a monic polynomial of degree one. It follows that $g(x) = (x - \lambda)$, for $\lambda \in \mathbb{R}$. Therefore, E_λ has dimension equal to one, since geometric multiplicity is less than or equal to algebraic multiplicity. Thus, for both eigenvalues of A, the geometric multiplicity equals the algebraic multiplicity, so A is diagonalizable. □

By the way: The proof did not need the fact that $\lambda = n$. Can you prove this?